

E_6 , the Group: The structure of $SL(3, \mathbb{O})$

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Abstract

We present the subalgebra structure of $\mathfrak{sl}(3, \mathbb{O})$, a particular real form of e_6 chosen for its relevance to particle physics and its close relation to generalized Lorentz groups. We use an explicit representation of the Lie group $SL(3, \mathbb{O})$ to construct the multiplication table of the corresponding Lie algebra $\mathfrak{sl}(3, \mathbb{O})$. Both the multiplication table and the group are then utilized to find various nested chains of subalgebras of $\mathfrak{sl}(3, \mathbb{O})$, in which the corresponding Cartan subalgebras are also nested where possible. Because our construction involves the Lie group, we simultaneously obtain an explicit representation of the corresponding nested chains of subgroups of $SL(3, \mathbb{O})$.

1 Introduction

The group E_6 has a long history of applications in physics [1, 2, 3], and is a candidate gauge group for a Grand Unified Theory [4]. A description of the group $E_{6(-26)}$ as $SL(3, \mathbb{O})$ was given in [5], generalizing the interpretation of $SL(2, \mathbb{O})$ as (the double cover of) $SO(9, 1)$ discussed in [6]. An interpretation combining spinor and vector representations of the Lorentz group in 10 spacetime dimensions was described in [7]. In this paper, we fill in some further details of the structure of $SL(3, \mathbb{O})$, in the process obtaining nested chains of subgroups that respect this Lorentzian structure.

Boosts	$B_{tz} \quad t \longleftrightarrow z \quad \mathbf{M} = \begin{pmatrix} \exp\left(\frac{\alpha}{2}\right) & 0 \\ 0 & \exp\left(-\frac{\alpha}{2}\right) \end{pmatrix}$ $B_{tx} \quad t \longleftrightarrow x \quad \mathbf{M} = \begin{pmatrix} \cosh\left(\frac{\alpha}{2}\right) & \sinh\left(\frac{\alpha}{2}\right) \\ \sinh\left(\frac{\alpha}{2}\right) & \cosh\left(\frac{\alpha}{2}\right) \end{pmatrix}$ $B_{tq} \quad t \longleftrightarrow q \quad \mathbf{M} = \begin{pmatrix} \cosh\left(\frac{\alpha}{2}\right) & -q \sinh\left(\frac{\alpha}{2}\right) \\ q \sinh\left(\frac{\alpha}{2}\right) & \cosh\left(\frac{\alpha}{2}\right) \end{pmatrix}$
Simple Rotations	$R_{xq} \quad x \longleftrightarrow q \quad \mathbf{M} = \begin{pmatrix} \exp\left(-\frac{q\alpha}{2}\right) & 0 \\ 0 & \exp\left(\frac{q\alpha}{2}\right) \end{pmatrix}$ $R_{xz} \quad x \longleftrightarrow z \quad \mathbf{M} = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & \sin\left(\frac{\alpha}{2}\right) \\ -\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}$ $R_{zq} \quad q \longleftrightarrow z \quad \mathbf{M} = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & q \sin\left(\frac{\alpha}{2}\right) \\ q \sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}$
Transverse Rotations	$R_{p,q} \quad p \longleftrightarrow q \quad \mathbf{M}_1 = -p \mathbf{I}_2$ $\mathbf{M}_2 = \left(\cos\left(\frac{\alpha}{2}\right) p + \sin\left(\frac{\alpha}{2}\right) q \right) \mathbf{I}_2$

Table 1: Finite octonionic Lorentz transformations. The group transformation is given by $\mathbf{X} \mapsto \mathbf{M}\mathbf{X}\mathbf{M}^\dagger$ for boosts and simple rotations, and by $\mathbf{X} \mapsto \mathbf{M}_2(\mathbf{M}_1\mathbf{X}\mathbf{M}_1^\dagger)\mathbf{M}_2^\dagger$ for transverse rotations. The parameters p and q are imaginary unit octonions.

We begin by reviewing the construction of both $SL(2, \mathbb{O})$ and $SL(3, \mathbb{O})$ at the group level in Section 2, then describe the construction of the Lie algebra $\mathfrak{sl}(3, \mathbb{O})$ in Section 3. In Section 4, we use this information to construct various chains of subgroups and subalgebras, some but not all of which are simple, and in Section 5 we discuss our results.

2 The Group

2.1 Lorentz transformations

A 2×2 Hermitian matrix

$$\mathbf{X} = \begin{pmatrix} t + z & x - q \\ x + q & t - z \end{pmatrix} \quad (1)$$

with $t, z, x \in \mathbb{R}$ and pure imaginary $q \in \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, is a representation of an $(m+1)$ -dimensional spacetime vector for $m+1 = ||\mathbb{K}|| + 2 \in \{3, 4, 6, 10\}$. In this setting, the squared Lorentzian norm of \mathbf{X} is given by $\det \mathbf{X}$. Lorentz transformations preserve $\det \mathbf{X}$ and must also preserve the Hermiticity of \mathbf{X} . Any Lorentz transformation can be described as the

Type 1	Type 2	Type 3
$\left(\begin{array}{c c} \mathbf{X} & \theta \\ \hline \theta^\dagger & \cdot \end{array} \right)$	$\left(\begin{array}{c c} \cdot & \theta^\dagger \\ \hline \theta & \mathbf{X} \end{array} \right)$	$\left(\begin{array}{c c c} \mathbf{X}_{2,2} & \theta_2 & \mathbf{X}_{2,1} \\ \hline \theta_2 & \cdot & \theta_1 \\ \hline \mathbf{X}_{1,2} & \theta_1 & \mathbf{X}_{1,1} \end{array} \right)$

Table 2: Three natural locations of a vector \mathbf{X} , a spinor θ , and a dual spinor θ^\dagger in $\mathcal{X} \in E_6$.

composition of maps of the form

$$\mathbf{X} \mapsto \mathbf{M}\mathbf{X}\mathbf{M}^\dagger \quad (2)$$

for certain *generators* \mathbf{M} . In the octonionic case, the determinant-preserving transformations of the form (2) constitute $SL(2, \mathbb{O})$, the (double cover of the) Lorentz group $SO(9, 1)$. We adopt the explicit set of generators constructed by Manogue and Schray [6], as given in Table 1.

An important feature of these transformations is that the *transverse rotations* between octonionic units require *nesting*; the lack of associativity prevents one from combining the given transformations of the form (2) into a single such transformation. We will return to this point in Section 3.

The exceptional Jordan algebra $\mathbf{H}_3(\mathbb{O})$, also known as the Albert algebra, consists of 3×3 octonionic Hermitian matrices under the Jordan product, and forms a 27-dimensional representation of E_6 [8], which is precisely the group that preserves the determinant of Jordan matrices; in this sense, $E_6 = SL(3, \mathbb{O})$. There are three natural ways to embed a 2×2 Hermitian matrix in a 3×3 Hermitian matrix, as illustrated in Table 2, which we refer to as *types*. Furthermore, $SL(2, \mathbb{O})$ sits inside $SL(3, \mathbb{O})$ under the identification

$$\mathbf{M} \mapsto \mathcal{M} = \left(\begin{array}{c|c} \mathbf{M} & 0 \\ \hline 0 & 1 \end{array} \right) \quad (3)$$

If we take $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$ to be of type 1, as per Table 2, then under the transformation

$$\mathcal{X} \mapsto \mathcal{M}\mathcal{X}\mathcal{M}^\dagger \quad (4)$$

we recover not only the *vector* transformation (2) on \mathbf{X} , but also the *spinor* transformation $\theta \mapsto \mathcal{M}\theta$ on the 2-component octonionic column θ .¹

We generalize this construction to all three types. We write M^1 (instead of \mathcal{M}) for the type 1 version of \mathcal{M} , as defined by (4). Then type 2 and 3 versions of \mathcal{M} can be obtained as

$$M^2 = \mathcal{T}M^1\mathcal{T}^\dagger \quad M^3 = \mathcal{T}M^2\mathcal{T}^\dagger \quad (5)$$

so that the group transformation

$$\mathcal{T} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in E_6 \quad (6)$$

¹Further discussion of “vectors” and “spinors” can be found in [7].

q	First pair	Second pair	Third pair
i	(j, k)	$(k\ell, j\ell)$	$(\ell, i\ell)$
j	(k, i)	$(i\ell, k\ell)$	$(\ell, j\ell)$
k	(i, j)	$(j\ell, i\ell)$	$(\ell, k\ell)$
$k\ell$	$(j\ell, i)$	$(j, i\ell)$	(k, ℓ)
$j\ell$	$(i, k\ell)$	$(i\ell, k)$	(j, ℓ)
$i\ell$	$(k\ell, j)$	$(k, j\ell)$	(i, ℓ)
ℓ	$(i\ell, i)$	$(j\ell, j)$	$(k\ell, k)$

Table 3: Quaternionic subalgebras chosen for A_q , G_q , and S_q .

cyclically permutes the 3 types. We discuss *type transformations* of the form (5) in more detail in Section 4.2

2.2 A new basis for transverse rotations

As outlined in [5, 7], E_6 can be viewed as the appropriate union of these 3 copies of $SO(9, 1, \mathbb{R}) = SL(2, \mathbb{O})$. But we have $3 \times 45 = 135$ elements, and we need to find a way to reduce this number to $|E_6| = 78$. We start by constructing a new basis for the transverse rotations in $SL(2, \mathbb{O})$.

Each transverse rotation $R_{p,q}$ listed in Table 1 rotates a single plane spanned by the orthogonal imaginary octonions p and q , and rotations of the 21 independent planes generate $SO(7)$. Since $G_2 \subset SO(7)$, we choose a basis for G_2 and extend it to $SO(7)$. For each basis octonion, say $q = i$, there are three pairs of basis octonions, in this case $\{j, k\}$, $\{k\ell, j\ell\}$, $\{\ell, i\ell\}$, which generate quaternionic subalgebras containing q . We have chosen the ordering of the pairs so that adding q leads to a right-handed, three-dimensional coordinate frame, and so that ℓ only appears (if at all) in the last pair. A choice of pairs for each basis octonion that satisfies these conditions is given in Table 3. We now define the combinations

$$\begin{aligned}
A_i(\alpha) &= R_{j,k}(\alpha) \circ R_{k\ell,j\ell}(-\alpha) \\
G_i(\alpha) &= R_{j,k}(\alpha) \circ R_{k\ell,j\ell}(\alpha) \circ R_{\ell,i\ell}(-2\alpha) \\
S_i(\alpha) &= R_{j,k}(\alpha) \circ R_{k\ell,j\ell}(\alpha) \circ R_{\ell,i\ell}(\alpha)
\end{aligned} \tag{7}$$

and use the conventions in Table 3 to similarly define A_q , G_q , and S_q for the remaining basis octonions.

As we discuss in more detail below, the 14 transformation of the form A_q and G_q generate the group G_2 , the 7 transformations A_q together with G_ℓ generate the subgroup $SU(3) \subset G_2$ which fixes ℓ , and all 21 of these transformations, which generate $SO(7)$, are orthogonal (but not normalized) at the Lie algebra level. We will use these properties to eliminate redundant group generators.

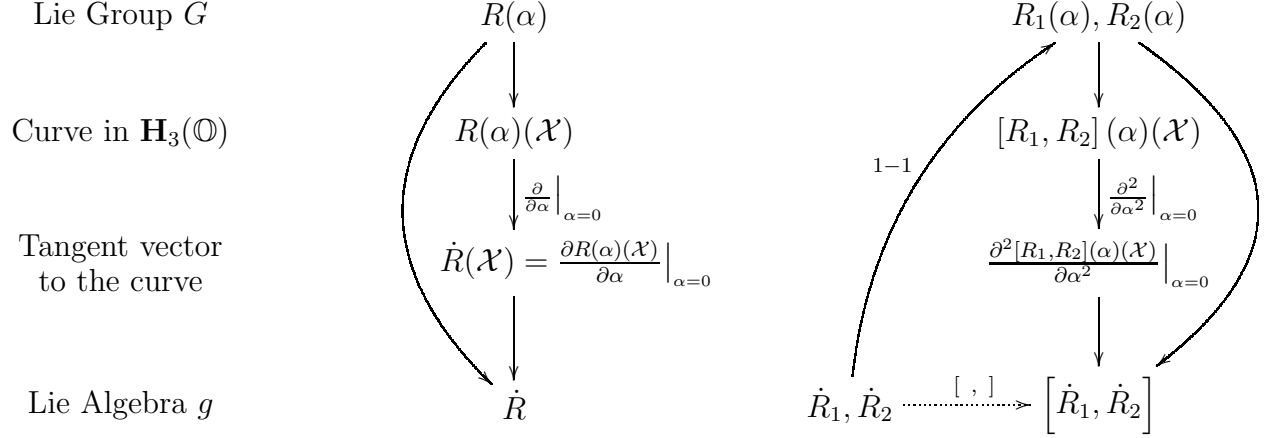


Figure 1: Calculating Lie algebra elements and their commutators.

3 The Lie Algebra

3.1 Constructing the algebra

We begin by associating each transformation in the Lie group with a vector in the Lie algebra. Each of the 135 transformations is a one-parameter curve in the group. Given a one-parameter curve $R(\alpha)$ in a classical Lie group, the traditional method for associating it with the Lie algebra generator \dot{R} is to find its tangent vector $\dot{R} = \left. \frac{\partial R(\alpha)}{\partial \alpha} \right|_{\alpha=0}$ at the identity element in the group. However, the transverse rotations are *nested*, that is, they involve more than one operation, and the lack of associativity prevents one from working with the group elements by themselves. Instead, we let our one-parameter transformations $R(\alpha)$ act on elements $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$, producing a curve $R(\alpha)(\mathcal{X})$ in $\mathbf{H}_3(\mathbb{O})$. We then define the Lie algebra element $\dot{R} \in \mathfrak{e}_6$ to be the map taking \mathcal{X} to the tangent vector at the identity to this curve in $\mathbf{H}_3(\mathbb{O})$. That is, we have the association indicated in Figure 1 between the group transformations and the tangent vectors.

We also use group orbits to construct the commutator of two tangent vectors. In the traditional approach to the classical matrix groups, the commutator of the tangent vectors \dot{R}_1 and \dot{R}_2 is defined as $[\dot{R}_1, \dot{R}_2] = \dot{R}_1 \dot{R}_2 - \dot{R}_2 \dot{R}_1$. However, we are working in $\mathbf{H}_3(\mathbb{O})$, not \mathfrak{e}_6 . To find the commutator of the Lie algebra elements \dot{R}_1 and \dot{R}_2 associated with curves $R_1(\alpha)(\mathcal{X})$ and $R_2(\alpha)(\mathcal{X})$, we create a new curve in $\mathbf{H}_3(\mathbb{O})$ defined by

$$[R_1, R_2](\alpha)(\mathcal{X}) = R_1\left(\frac{\alpha}{2}\right) \circ R_2\left(\frac{\alpha}{2}\right) \circ R_1\left(-\frac{\alpha}{2}\right) \circ R_2\left(-\frac{\alpha}{2}\right)(\mathcal{X}) \quad (8)$$

where \circ denotes composition. This new path is not a one-parameter curve, and its first derivative is identically zero at $\alpha = 0$, but its second derivative is tangent to the curve $[R_1, R_2](\alpha)(\mathcal{X})$ at $\alpha = 0$. Therefore, we define the commutator of \dot{R}_1 and \dot{R}_2 by the following

action on $\mathbf{H}_3(\mathbb{O})$

$$\left[\dot{R}_1, \dot{R}_2\right](\mathcal{X}) = \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} [R_1, R_2](\alpha)(\mathcal{X}) \Big|_{\alpha=0} \quad (9)$$

which agrees with the usual definition for matrix Lie groups [9]. Our construction of the commutator is summarized in Figure 1.

Since we are using the local action of $SL(3, \mathbb{O})$ on $\mathbf{H}_3(\mathbb{O})$ to give a homomorphic image of $\mathfrak{sl}(3, \mathbb{O})$, our construction does not lead to a readily available exponential map giving the group element corresponding to $[\dot{R}_1, \dot{R}_2]$. In particular, we are not always able to find the one-parameter curve whose tangent vector is $[\dot{R}_1, \dot{R}_2]$.

3.2 Linear dependencies

We shall now give the dependencies among the group transformations by using linear dependencies among the Lie algebra elements. In doing so, we will indicate which transformations can be eliminated, leaving our preferred basis for the group $SL(3, \mathbb{O})$ and the algebra $\mathfrak{sl}(3, \mathbb{O})$. Since we are using a homomorphic image of the Lie algebra $\mathfrak{sl}(3, \mathbb{O})$, we check that the indicated dependencies actually do provide dependencies among the group transformations.

We begin with the transverse rotations. Among the 21 transformations A_q , G_q , and S_q of each type, direct computation shows that

$$\dot{A}_q^1 = \dot{A}_q^2 = \dot{A}_q^3 \quad \dot{G}_q^1 = \dot{G}_q^2 = \dot{G}_q^3 \quad (10)$$

for each basis octonion q . That is, the transformations A_q and G_q are type independent, allowing us to drop the type designation and simply write \dot{A}_q and \dot{G}_q . These fourteen transformations generate $G_2 = \text{Aut}(\mathbb{O})$, which is the smallest of the exceptional Lie groups. We refer to the type independence of these transformations as *strong triality*. When added to the fourteen G_2 transformations, the seven transformations S_q^a produce a basis for the $SO(7)$ of type a , with $a = 1, 2, 3$. However, the transformations S_q^a are not independent, since

$$\dot{S}_q^1 + \dot{S}_q^2 + \dot{S}_q^3 = 0 \quad (11)$$

Hence, the union of any two of the $SO(7)$ subgroups contains the third. In particular, we may use the group transformations generated by S_q^a of type 1 and type 2 to generate the type 3 transformations generated by S_q^3 . These linear dependences have reduced our $3 \times 21 = 63$ transverse rotations by $28 + 7 = 35$, trimming our original 135 transformations down to 100.

Turning to $SO(8)$, we have the relations

$$\begin{aligned} 0 &= \dot{R}_{xq}^1 + \dot{R}_{xq}^2 + \dot{R}_{xq}^3 \\ \dot{R}_{xq}^2 &= -\frac{1}{2} \dot{R}_{xq}^1 - \frac{1}{2} \dot{S}_q^1 \\ \dot{S}_q^2 &= \frac{3}{2} \dot{R}_{xq}^1 - \frac{1}{2} \dot{S}_q^1 \end{aligned} \quad (12)$$

which allow us to eliminate a further 21 transformations. We have in fact expressed all $SO(8)$ transformations of types 2 and 3 in terms of $SO(8)$ transformations of type 1; in this sense, there is only one $SO(8)!$ Again, this is a result of *triality*.

Having reduced the 135 transformations to 100 and then by another 21 to 79, we are left with 52 rotations, which preserve the trace of $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$, and which form the Lie group $F_4 = SU(3, \mathbb{O})$. Among the remaining 27 boosts, we expect only one additional linear dependency, which turns out to be

$$\dot{B}_{tz}^1 + \dot{B}_{tz}^2 + \dot{B}_{tz}^3 = 0 \quad (13)$$

which we use to eliminate \dot{B}_{tz}^2 and \dot{B}_{tz}^3 in favor of the combination $\dot{B}_{tz}^2 - \dot{B}_{tz}^3$. The resulting 78 Lie algebra elements are indeed independent, and turn out to be orthogonal (but not normalized) with respect to the Killing form.

We have therefore constructed both the group $E_6 = SL(3, \mathbb{O})$, and its Lie algebra $\mathfrak{e}_6 = \mathfrak{sl}(3, \mathbb{O})$; the complete commutation table for $\mathfrak{sl}(3, \mathbb{O})$ can be found online at [10]. In retrospect, the counting is easy: There is one $SO(8)$ (28 elements), 3 types of each of the remaining elements of $SO(9)$ (24 elements, yielding F_4), and 3 types of the 9 boosts, with one final dependency, yielding 26 boosts in all.

Our basis can be simplified slightly by noticing that (12) implies

$$\dot{S}_q^1 = \dot{R}_q^3 - \dot{R}_q^2 \quad (14)$$

where the operations on the RHS commute. Thus, the diagonal *phase* transformations S_q^1 can in fact be constructed *without* nesting, which however is essential for the G_2 transformations A_q and G_q . This provides another way to count the basis of \mathfrak{e}_6 : There are 64 independent trace-free 3×3 octonionic matrices, $24 + 14 = 38$ of which are anti-Hermitian (infinitesimal rotations), and $24 + 2 = 26$ of which are Hermitian (boosts), together with the 14 nested transformations making up \mathfrak{g}_2 , for a total of 78 independent elements in \mathfrak{e}_6 [8].

We can further identify the 6 elements

$$C = \{\dot{B}_{tz}^1, \dot{B}_{tz}^2 - \dot{B}_{tz}^3, \dot{R}_{x\ell}^1, \dot{A}_\ell, \dot{G}_\ell, \dot{S}_\ell^1\} \quad (15)$$

as a commuting set, and therefore a preferred (orthogonal) basis for the Cartan subalgebra h . We call these basis elements the *Cartan elements* of \mathfrak{e}_6 .

4 Subalgebra Chains

4.1 Basic subalgebra chains

We begin with a discussion of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. Our basis selects a preferred $\mathfrak{su}(3)$ subalgebra of \mathfrak{g}_2 , namely the \mathfrak{g}_2 transformations which fix the preferred complex subalgebra of \mathbb{O} generated by ℓ . Günaydin denotes the corresponding $SU(3)$ subgroup of G_2 as $SU(3)^C$ [11]; we prefer to use the name $\mathfrak{su}(3)_\mathbb{C}$ for this subalgebra. Explicitly, we have

$$\mathfrak{su}(3)_\mathbb{C} = \langle \dot{A}_i, \dots, \dot{A}_{i\ell}, \dot{A}_\ell, \dot{G}_\ell \rangle \quad (16)$$

which is also a subalgebra of the (type 1, say) $\mathfrak{so}(6) \subset \mathfrak{so}(7)$ that fixes ℓ .

Through a conventional choice of A_ℓ , our basis also selects a preferred quaternionic subalgebra of \mathbb{O} , generated by $\{k, k\ell, \ell\}$, and a preferred subalgebra $\mathfrak{su}(2)_{\mathbb{H}} \subset \mathfrak{su}(3)_{\mathbb{C}}$ that fixes this quaternionic subalgebra, namely

$$\mathfrak{su}(2)_{\mathbb{H}} = \langle \dot{A}_k, \dot{A}_{k\ell}, \dot{A}_\ell \rangle \quad (17)$$

Extending to $\mathfrak{so}(7)$, there is clearly an $\mathfrak{so}(4)$ that fixes \mathbb{H} ; we have

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{su}(2)_{\mathbb{H}} \oplus \langle \dot{G}_k - \dot{S}_k^1, \dot{G}_{k\ell} - \dot{S}_{k\ell}^1, \dot{G}_\ell - \dot{S}_\ell^1 \rangle \quad (18)$$

as can be seen by studying Table 3. Another interesting $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(7)$ is the complement of this $\mathfrak{so}(4)$, an orthogonal basis for which is given by the combinations $\dot{G}_q + 2\dot{S}_q^1$ for $q \in \text{Im}\mathbb{H}$.

We can use our particular choice of basis for the Lie algebra \mathfrak{e}_6 to identify two separate $SO(n)$ subgroup structures within the Lie group E_6 . Figure 2 shows the $SO(n)$ subgroup chain of $SO(9, 1)$ of type 1 in $SL(3, \mathbb{O})$, while Figure 3 shows the three $SO(9)$ subgroup chains of F_4 within E_6 . In both subgroup structures, there is only one $SO(8)$. While $G_2 \subset SO(7)$, it is not a subset of $SO(6)$ in Figure 2. Hence, we omit G_2 from Figure 2, but include it in Figure 3 since our preferred basis for $SO(7)$ includes a basis for G_2 . The figures indicate which Cartan element is added to a group when it is expanded to a larger group, as well as giving the classification of the corresponding Lie algebra.

4.2 Type transformations

The *discrete type transformation* (5) induced by \mathcal{T} cyclically permutes Lorentz transformations of types 1, 2, or 3. We have

$$\mathcal{T}^3 = \mathcal{I} \quad (19)$$

$$\mathcal{T}^\dagger = \mathcal{T}^{-1} \quad (20)$$

where \mathcal{I} is the 3×3 identity matrix, and $\mathcal{T} \in SL(3, \mathbb{O})$, since

$$\det(\mathcal{T}\mathcal{X}\mathcal{T}^\dagger) = \det(\mathcal{X}) \quad (21)$$

for $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$. Although \mathcal{T} is not one of our elementary group transformations, there are numerous identities of the form

$$\begin{aligned} \mathcal{T} &= R_{xz}^1(-\pi) \circ R_{xz}^2(-\pi) \\ \mathcal{T} &= R_{xz}^2(\pi) \circ R_{xz}^1(\pi) \circ R_{xz}^2(\pi) \circ R_{xz}^1(\pi) \\ \mathcal{T} &= R_{xz}^1(\pi) \circ R_{xz}^3(\pi) \circ R_{xz}^2(\pi) \circ R_{xz}^1(\pi) \end{aligned} \quad (22)$$

These expressions make clear that $\mathcal{T} \in SL(3, \mathbb{O})$. Furthermore, each of these expressions may be expanded into a (different) *continuous type transformation* $\mathcal{T}(\alpha) \in SL(3, \mathbb{O})$ by letting the single fixed angle (π or $-\pi$) become arbitrary. The resulting transformations are *not* one-parameter subgroups of $SL(3, \mathbb{O})$, but they do connect transformations of different types. We are therefore led to explore subgroups of $SL(3, \mathbb{O})$ that contain these (real!) type transformations, although it suffices to consider subgroups containing \mathcal{T} itself.

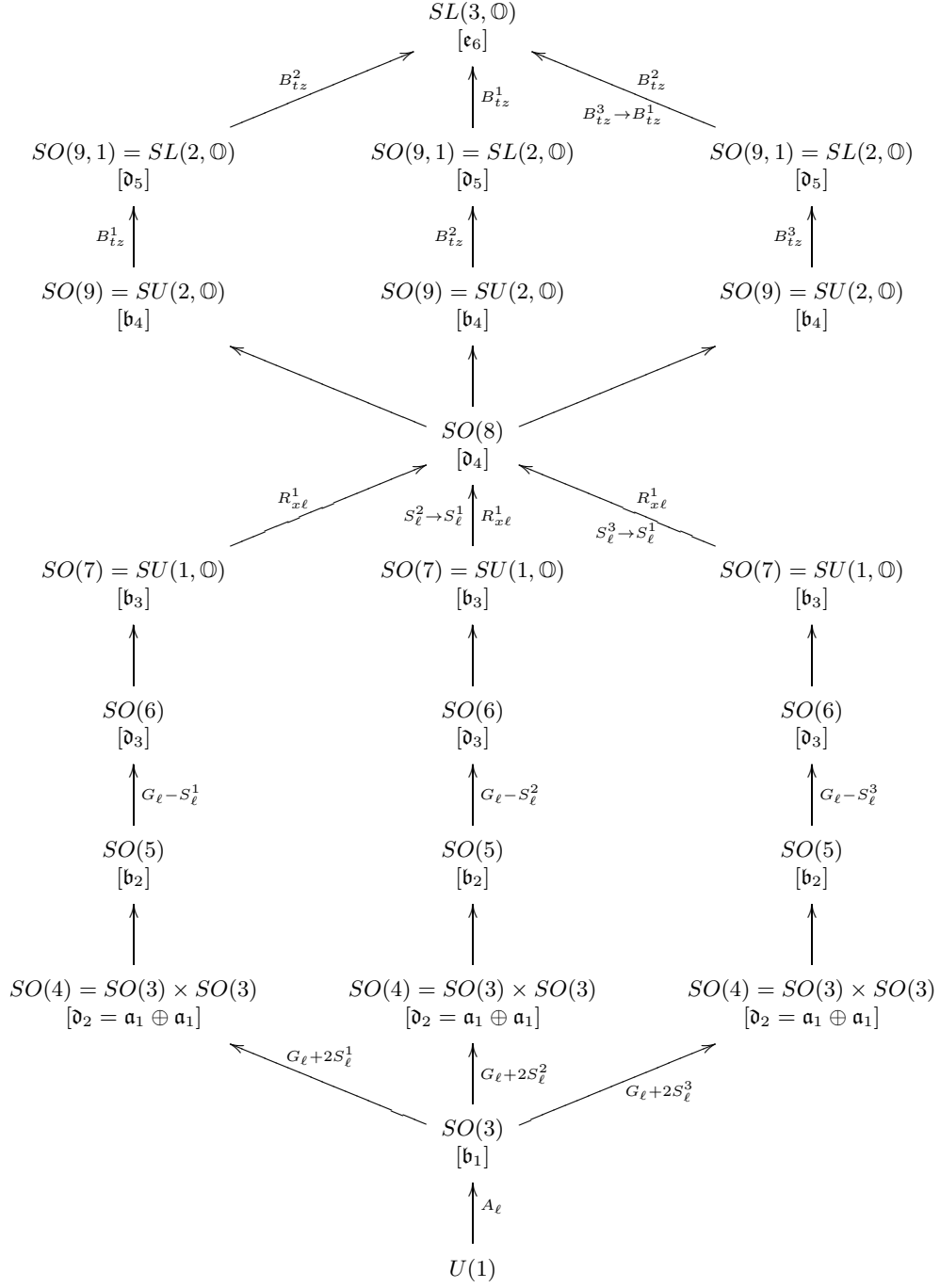


Figure 2: Chain of subgroups $SO(n) \subset SO(9, 1) \subset SL(3, \mathbb{O})$.

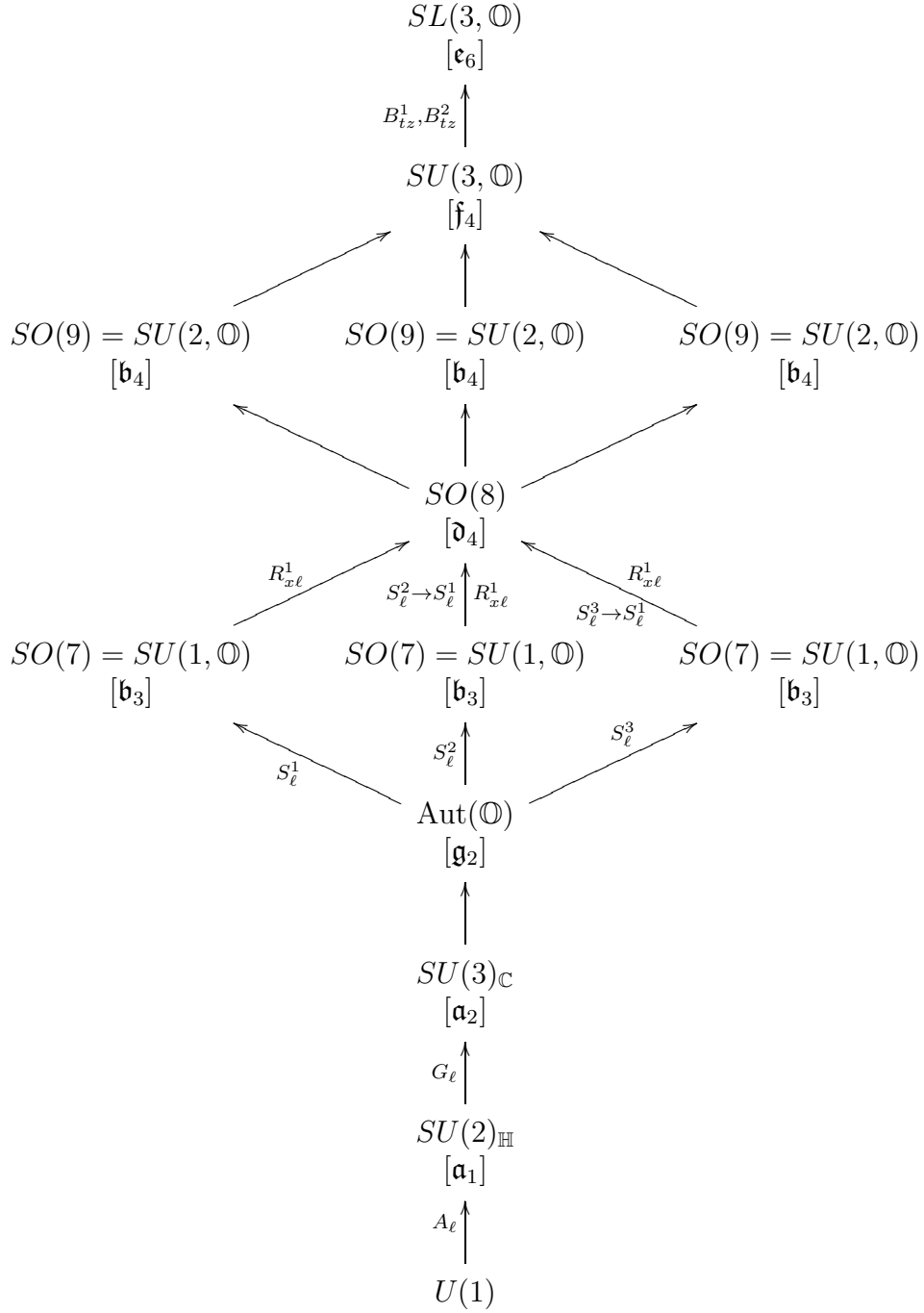


Figure 3: Chain of subgroups $SU(3)^C \subset G_2 \subset SO(8) \subset F_4 \subset E_6$.

4.3 Type-independent subgroups

We list here some important groups which contain type transformations. The standard representation of $SO(3, \mathbb{R})$ is the group

$$SO(3, \mathbb{R})_s = \langle R_{xz}^1, R_{xz}^2, R_{xz}^3 \rangle \quad (23)$$

This group obviously contains \mathcal{T} , as does the standard representation

$$SL(3, \mathbb{R})_s = \langle R_{xz}^1, R_{xz}^2, R_{xz}^3, B_{tz}^1, B_{tz}^2, B_{tx}^1, B_{tx}^2, B_{tx}^3 \rangle \quad (24)$$

of $SL(3, \mathbb{R})$. Using ℓ as our preferred complex unit, we have the standard representations

$$SU(3, \mathbb{C})_s = \langle R_{xz}^1, R_{xz}^2, R_{xz}^3, R_{x\ell}^1, R_{x\ell}^2, R_{z\ell}^1, R_{z\ell}^2, R_{z\ell}^3 \rangle \quad (25)$$

of $SU(3, \mathbb{C})$, and

$$SL(3, \mathbb{C})_s = SU(3, \mathbb{C})_s \cup \langle B_{tz}^1, B_{tz}^2, B_{tx}^1, B_{tx}^2, B_{tx}^3, B_{t\ell}^1, B_{t\ell}^2, B_{t\ell}^3 \rangle \quad (26)$$

of $SL(3, \mathbb{C})$. These four groups are important because they contain the type transformation \mathcal{T} . If, for instance, some type 1 transformation R^1 is in a group G that has one of these groups as a subgroup, then G must also contain the corresponding type 2 and 3 transformations R^2 and R^3 ; we say that G is *type independent*.

The standard representations $SO(3, \mathbb{R})_s$ and $SU(3, \mathbb{C})_s$ differ from our preferred representations $SO(3)_{\mathbb{H}} = SU(2)_{\mathbb{H}}$ and $SU(3)_{\mathbb{C}}$, which are subgroups of G_2 . For instance, the groups $SU(3, \mathbb{C})_s$ and $SU(3)_{\mathbb{C}}$ are both type independent, but in $SU(3, \mathbb{C})_s$ the transformations R^1, R^2 and R^3 are distinct while in $SU(3)_{\mathbb{C}}$ the three transformations are equal; $SU(3)_{\mathbb{C}}$ does not contain \mathcal{T} , nor does it need to.

We use the type transformation \mathcal{T} to provide insight into the structure of the Lie algebra $\mathfrak{sl}(3, \mathbb{O})$. The algebras \mathfrak{g} in the left column of Figure 4 are subalgebras of the type 1 copy of $\mathfrak{sl}(2, \mathbb{O})$, while each algebra \mathfrak{g}' in the right column is the largest subalgebra of $\mathfrak{sl}(3, \mathbb{O})$ such that $\mathfrak{g} \oplus \mathfrak{g}'$ is still simple. When we restrict \mathfrak{g} to a smaller subalgebra of $\mathfrak{sl}(2, \mathbb{O})$, it is sometimes possible to expand the type-independent subalgebra \mathfrak{g}' to a larger subalgebra of $\mathfrak{sl}(3, \mathbb{O})$. Each arrow in the diagram indicates inclusion, and a similar diagram holds for the corresponding subgroups of $SL(3, \mathbb{O})$.

4.4 Reduction of \mathbb{O} to \mathbb{H} , \mathbb{C} , and \mathbb{R}

We can also find subalgebras of \mathfrak{e}_6 by restricting our generators to be quaternionic, complex, or real.

Our preferred quaternionic subalgebra of \mathbb{O} is $\mathbb{H} = \langle 1, k, k\ell, \ell \rangle$, so we discard transformations involving $i, j, i\ell$, or $j\ell$. We therefore discard $3 \times 4 = 12$ boosts, $3 \times 4 = 12$ simple rotations involving z , and 4 simple rotations involving x — but we must add back in 3 rotations involving x of type 2, since we can no longer use the middle relation in (12) to eliminate them. Turning to the transverse rotations, we need only consider transformations

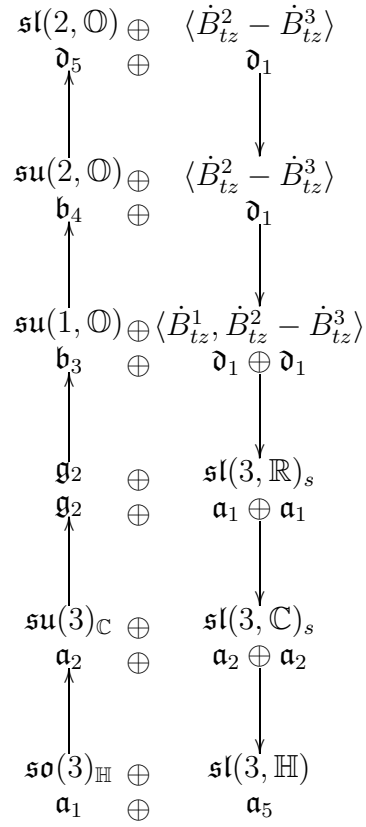


Figure 4: Type-dependent and type-independent subalgebras of \mathfrak{e}_6 .

of type 1, and, as discussed in Section 4.1 (or after studying Table 3), we see that we must retain only the combinations $\dot{G}_q - \dot{S}_q^1$ for $q \in \text{Im}\mathbb{H}$, thus discarding the remaining $21 - 3 = 18$ elements of $\mathfrak{so}(7)$. We are left with $52 - 34 + 3 = 21$ rotations, and $26 - 12 = 14$ boosts.

The 21 compact generators form the algebra $\mathfrak{su}(3, \mathbb{H})$, a real form of $\mathfrak{c}_3 = \mathfrak{sp}(6)$, while all 35 together form $\mathfrak{sl}(3, \mathbb{H})$, a real form of $\mathfrak{a}_5 = \mathfrak{su}(6, \mathbb{C})$. Restricting only to type 1 transformations, we obtain 10 rotations and 5 boosts, thus reducing $\mathfrak{sl}(3, \mathbb{H})$ to $\mathfrak{sl}(2, \mathbb{H}) = \mathfrak{so}(5, 1)$, a real form of $\mathfrak{d}_3 = \mathfrak{so}(6)$, and $\mathfrak{su}(3, \mathbb{H})$ to $\mathfrak{su}(2, \mathbb{H}) = \mathfrak{so}(5)$, a real form of $\mathfrak{c}_2 = \mathfrak{sp}(4)$.

Furthermore, the subalgebra $\mathfrak{so}(3)_{\mathbb{H}} = \langle \dot{A}_k, \dot{A}_{k\ell}, \dot{A}_\ell \rangle$ fixes \mathbb{H} . Thus, for each of the above subalgebras \mathfrak{g} , we have $\mathfrak{g} \oplus \mathfrak{so}(3)_{\mathbb{H}} \in \mathfrak{sl}(3, \mathbb{O})$. In particular, $\mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{so}(3)_{\mathbb{H}}$ is therefore a subalgebra of $\mathfrak{sl}(3, \mathbb{O})$.

When restricting \mathbb{O} to our preferred complex subalgebra $\mathbb{C} = \langle 1, \ell \rangle$, we obtain the classical Lie algebras $\mathfrak{su}(3, \mathbb{C})_s$ and $\mathfrak{sl}(3, \mathbb{C})_s$ as previously discussed. As there is only one octonionic unit used to form \mathbb{C} , we do not need to use any of the transformations from $SO(7)$, so we have 8 rotations and 8 boosts. Using all 16 transformations gives $\mathfrak{sl}(3, \mathbb{C})_s$, a real form of $\mathfrak{a}_2 \oplus \mathfrak{a}_2 = \mathfrak{su}(3, \mathbb{C}) \oplus \mathfrak{su}(3, \mathbb{C})$ with 8 boosts, whereas we obtain $\mathfrak{su}(3, \mathbb{C})_s$ by using only the 8 compact generators. Further restricting to the type 1 transformations reduces these two algebras to $\mathfrak{sl}(2, \mathbb{C})_s = \mathfrak{so}(3, 1)_s$ and $\mathfrak{su}(3, \mathbb{C})_s$, which are real forms of $\mathfrak{d}_2 = \mathfrak{su}(2, \mathbb{C}) \oplus \mathfrak{su}(2, \mathbb{C})$ and $\mathfrak{a}_1 = \mathfrak{su}(3, \mathbb{C})$.

When we restrict $\mathfrak{sl}(3, \mathbb{C})_s$ to $\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{sl}(2, \mathbb{O})$ (of type 1, say), the smaller algebra no longer contains the type transformation \mathcal{T} , but it does involve the octonionic direction ℓ . Thus, $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(6)$, where $\mathfrak{so}(6) \subset \mathfrak{so}(7)$ fixes ℓ , is also a subalgebra of $\mathfrak{sl}(2, \mathbb{O}) \subset \mathfrak{sl}(3, \mathbb{O})$.

Finally, by restricting to real transformations, we are left with 3 rotations and 5 boosts, which is a real form of $\mathfrak{a}_2 = \mathfrak{su}(3, \mathbb{C})$ with 5 non-compact elements. This algebra may be further restricted to either $\mathfrak{so}(3, \mathbb{R})_s$, whose group contains the type transformation, or $\mathfrak{so}(2, 1)_s$, which is a type 1 non-compact form of $\mathfrak{a}_1 = \mathfrak{so}(3, \mathbb{R})$.

The above discussion of the result of restricting $\mathfrak{sl}(3, \mathbb{O})$ to $\mathfrak{sl}(n, \mathbb{K})$ for $n = 1, 2, 3$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is summarized in Figure 5. For each algebra \mathfrak{g} in Figure 5, we then list in Figure 6 the maximal subalgebra \mathfrak{g}' of \mathfrak{e}_6 such that $\mathfrak{g} \oplus \mathfrak{g}' \in \mathfrak{sl}(3, \mathbb{O})$. Here, $\mathfrak{so}(6)$ again denotes the subalgebra of type 1 which permutes $\{i, j, k, k\ell, j\ell, i\ell\}$ but fixes ℓ . Although $\mathfrak{so}(6) \not\subset \mathfrak{g}_2$, we do have $\mathfrak{su}(3)_{\mathbb{C}} \subset \mathfrak{so}(6)$. We also write $\mathfrak{u}(-1)$ for the non-compact real representation of \mathfrak{d}_1 generated by $\dot{B}_{tz}^2 - \dot{B}_{tz}^3$, which is discussed further in the next section. Again, similar diagrams can be drawn for the corresponding subgroups of $SL(3, \mathbb{O})$.

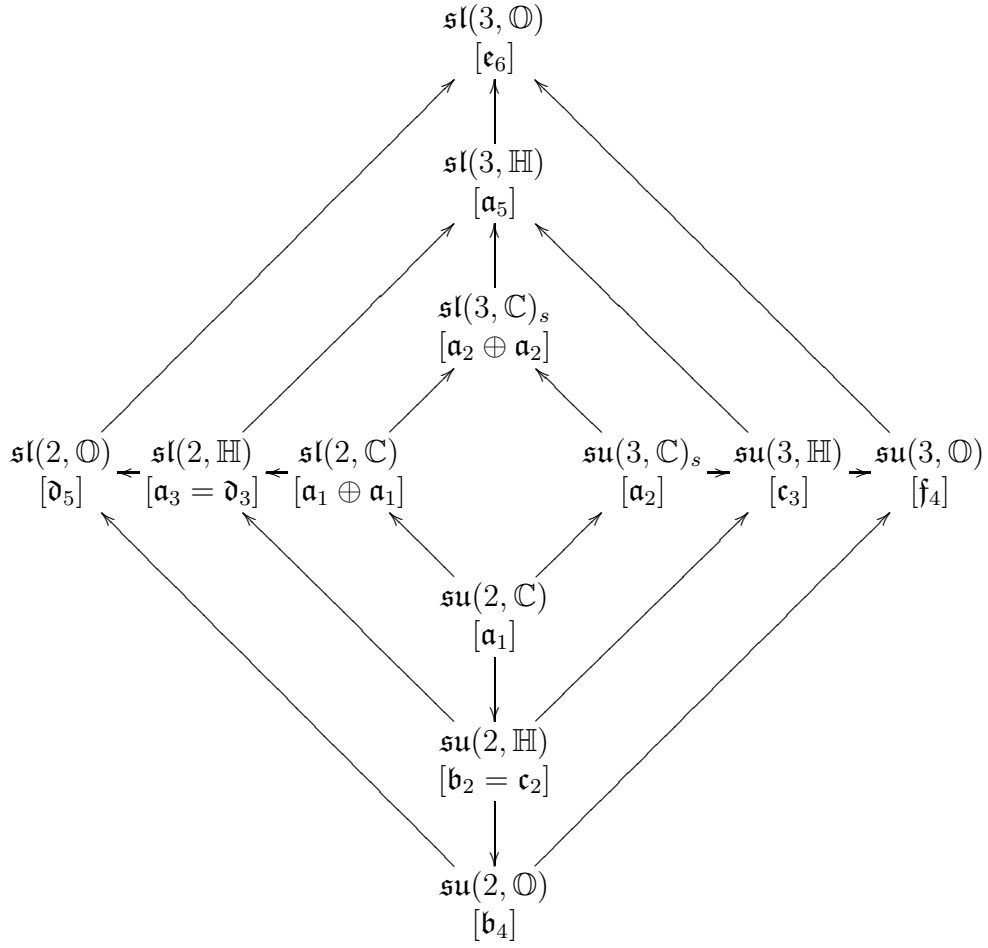


Figure 5: Subalgebras $\mathfrak{sl}(n, \mathbb{K})$ and $\mathfrak{su}(n, \mathbb{K})$ of $\mathfrak{sl}(3, \mathbb{O})$.

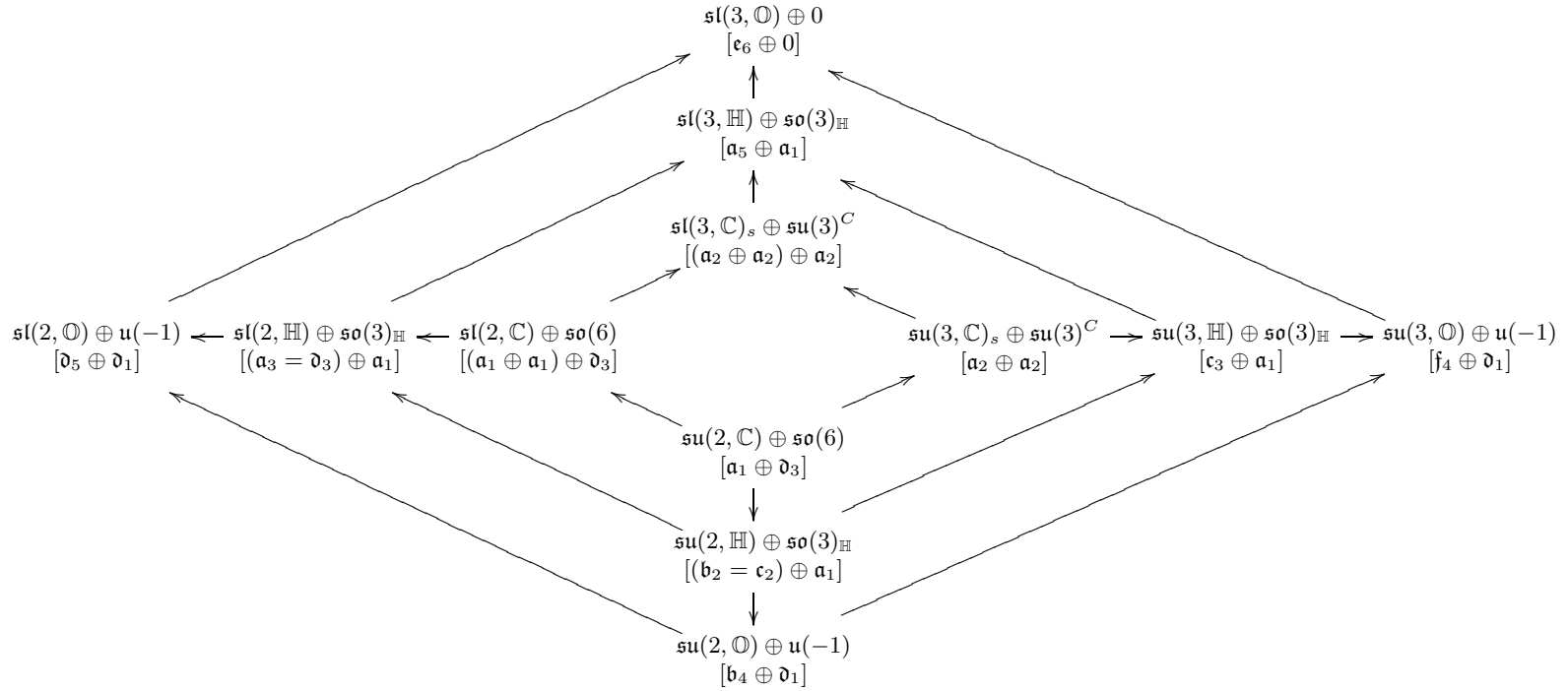


Figure 6: Subalgebras $\mathfrak{sl}(n, \mathbb{K}) \oplus \mathfrak{g}'$ and $\mathfrak{su}(n, \mathbb{K}) \oplus \mathfrak{g}'$ of $\mathfrak{sl}(n, \mathbb{O})$.

4.5 Subalgebras fixing type

Having just considered the subalgebras of \mathfrak{g}_2 , and hence of \mathfrak{e}_6 , that leave invariant a preferred complex or quaternionic subalgebra of \mathbb{O} , we now ask what subalgebra of \mathfrak{e}_6 fixes all type 1 elements, that is, which transformations leave \mathbf{X} alone in the first decomposition of $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$ shown in Table 2. This subalgebra, which we will call $\text{stab}(I)$, turns out to be quite different from any of the others discussed previously.

Clearly, no transformation in (type 1) $\mathfrak{sl}(2, \mathbb{O})$ will be in $\text{stab}(I)$. We therefore seek transformations of types 2 and 3. Direct computation shows that certain *null rotations* will do the job. Each of the 6 vector spaces defined by

$$b_{\pm}^a = \langle \dot{B}_{tx}^a \mp \dot{R}_{xz}^a, \dot{B}_{tq}^a \pm \dot{R}_{zq}^a \rangle \quad (27)$$

is in fact an abelian subalgebra of $\mathfrak{sl}(3, \mathbb{O})$, and in each case the given basis elements are null according to the Killing form — the Killing form is in fact identically zero on each of these subalgebras. Each of these subalgebras fixes all elements of a particular type; we have

$$\text{stab}(I) = b_+^2 \oplus b_-^3 \quad (28)$$

with cyclic permutations holding for $\text{stab}(II)$ and $\text{stab}(III)$.

Since $\text{stab}(I)$ contains no elements of (type 1) $\mathfrak{sl}(2, \mathbb{O})$, we expect that $\mathfrak{sl}(2, \mathbb{O}) \oplus \text{stab}(I)$ will be a $45 + 16 = 61$ -dimensional subalgebra of $\mathfrak{sl}(3, \mathbb{O})$. Checking commutators, this turns out to be correct, but with an unexpected surprise: $\text{stab}(I)$ is an ideal of $\mathfrak{sl}(2, \mathbb{O}) \oplus \text{stab}(I)$, so this subalgebra is neither simple nor semisimple.

If we further define

$$\text{stab}(I)^\perp = b_-^2 \oplus b_+^3 \quad (29)$$

to be the 16 null rotations of types 2 and 3 that are not in $\text{stab}(I)$, then we have the intriguing decomposition

$$\mathfrak{sl}(3, \mathbb{O}) = \mathfrak{sl}(2, \mathbb{O}) \oplus \text{stab}(I) \oplus \text{stab}(I)^\perp \oplus \mathfrak{u}(-1) \quad (30)$$

with $\mathfrak{u}(-1)$ again denoting the non-compact real representation of \mathfrak{d}_1 generated by $\dot{B}_{tz}^2 - \dot{B}_{tz}^3$.

We can now easily determine the subalgebras of \mathfrak{e}_6 that, say, leave \mathbb{H} or \mathbb{C} in type 1 elements invariant. All we have to do is combine the relevant subalgebra of $\mathfrak{sl}(2, \mathbb{O})$ — in this case $\mathfrak{su}(2)_{\mathbb{H}}$ or $\mathfrak{su}(3)_{\mathbb{C}}$, respectively — with $\text{stab}(I)$. Each such algebra, here $\mathfrak{su}(2)_{\mathbb{H}} \oplus \text{stab}(I)$ and $\mathfrak{su}(3)_{\mathbb{C}} \oplus \text{stab}(I)$ is a subalgebra of \mathfrak{e}_6 which, however, is neither simple nor semisimple. Two further examples are the 52-dimensional subalgebras $\mathfrak{su}(2, \mathbb{O}) \oplus \text{stab}(I)$, which fixes (type 1) t , and $\mathfrak{so}(8, 1)_{\ell} \oplus \text{stab}(I)$, where $\mathfrak{so}(8, 1)_{\ell}$ fixes (type 1) ℓ (and therefore does not contain g_2).

5 Conclusion

In this paper, we have given an explicit description of the subgroup structure of $SL(3, \mathbb{O})$, based on the “type” structure inherent in the embedding of $SL(2, \mathbb{O})$ in $SL(3, \mathbb{O})$, and on the structure of $SL(2, \mathbb{O})$ itself. In the process, we have provided explicit realizations of some of the remarkable properties of G_2 . The internal structure of G_2 , such as the $SU(3)$ and $SU(2)$ subgroups fixing either a complex or quaternionic subalgebra, may be especially relevant to attempts to use $SL(3, \mathbb{O})$ to describe fundamental particles, as discussed further in [7]. Furthermore, we have seen explicitly how G_2 is preserved under triality, as discussed in [5]. Finally, we have constructed the groups leaving the type structure invariant, which we suspect may play a prominent role in describing the interactions of fundamental particles.

However, the story is only partially complete. There are other interesting subgroups of $SL(3, \mathbb{O})$, closely related to the 4 other real forms of E_6 . In particular, we have not yet identified any of the C_4 subgroups of E_6 . In other work [12], we extend, and in a sense complete, the present investigation by constructing and discussing chains of subgroups adapted to these other subgroups. We hope that the resulting maps of $E_{6(-26)}$ will prove useful in further attempts to apply the exceptional groups to nature.

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